J. Fluid Mech. (1972), *vol.* 52, *part* 2, *pp.* 321–344
Printed in Great Britain

Variational properties of steady fall in Stokes flow

By H. F. WEINBERGER

(Received 16 June 1971 and in revised form 18 October 1971)

It is shown that for a given body and a given orientation $\mathbf{g}$ there is always a position of the centre of mass which produces a stable falling motion in a very viscous fluid with $\mathbf{g}$ vertical and, in general, with a spin about the vertical axis. The corresponding terminal settling speed is bounded by means of several variational principles.

Relations between the terminal speeds for falls with different downward directions and between the terminal speed and the geometry of the body are deduced. In particular, it is proved that for a large class of slender bodies the first approximation to the drag obtained from the slender-body theory of Burgers (1938) is correct. It follows that the ratio of the terminal speeds for falls with the long axis vertical and horizontal is near two.

1. Introduction

It was shown experimentally by Taylor (1967, 1969) that, if a slender convex axially symmetric body is allowed to fall in a viscous fluid, its terminal speed when it falls along its axis of symmetry is about twice its terminal speed when it falls with its axis of symmetry in a horizontal direction. Taylor gave a heuristic argument using the slender-body theory of Burgers (1938) to support the conjecture that this result should be true for any sufficiently slender convex axially symmetric body. In this paper we shall establish this and other results by means of a quadratic variational theory.

In order to treat the problem of the steady fall of a general body in the Stokes regime it is first necessary to formulate physically and mathematically correct boundary-value problems. Two such problems are formulated in § 2. In problem 1 the downward orientation of the body is prescribed and one tries to determine a position of the centre of mass for which the body has a steady falling motion with the given direction downward. In problem 2 the centre of mass is given and one wishes to find a downward orientation for which a steady falling motion exists. In § 3 we show that problem 1 has a unique solution and that problem 2 always has at least one solution. Moreover, we characterize the solution of problem 1 by several variational principles for the terminal speed.

A minimum energy dissipation principle for a Stokes flow with given velocity on the boundary was introduced by Helmholtz (1868) and proved by Korteweg (1883). Complementary energy principles for such problems were given by Hill & Power (1956) and by Kearsley (1960). These principles were generalized by Keller, Rubenfeld & Molyneux (1967) to a fluid containing rigid particles with given instantaneous positions and orientations which are settling in the fluid under the action of given forces and torques, and also containing inclusions of a different fluid in given instantaneous positions. A further generalization to include surface tension on the boundaries of the fluid bubbles was given by Skalak (1970). The forces and torques on each rigid particle must be prescribed for the variational principles of Keller, Rubenfeld & Molyneux and Skalak. Therefore, these principles cannot be applied to problem 1, in which the position of the centre of mass, and hence the gravitational torque, is to be determined. Therefore our variational principles do not follow from the earlier ones. Of course, all these variational principles fit into the framework of the general quadratic variational theory. (See, e.g. Diaz 1951; Mikhlin 1957; Synge 1957; Weinberger 1961.)

It is interesting to note that we derive two non-equivalent minimum principles as well as a maximum principle. While these principles could be used for numerical computation by means of the Rayleigh–Ritz method, we shall use them only to obtain general properties of the terminal settling speed. In particular, we show in §3 (theorem 5) that for a given net weight the settling speed decreases if the body is enlarged. We also obtain a sharp bound for the settling speed in terms of the electrostatic capacity of the body. Our results do not require the body to have a smooth boundary, but they do assume that the electrostatic capacity of the body is positive. Thus, they apply to a disk but not to a line segment. In fact, if we consider a line segment as a limit of thin ellipsoids, we see that its terminal settling speed should be infinite, so that the corresponding Stokes-flow problem has no solution.

In §4 we show that if the centre of mass is sufficiently low the steady falling motion is the limit in the quasi-steady Stokes regime of any falling motion except for the unstable motion in which the body falls upside down. In establishing this result we use the translation, rotation and coupling tensors of Brenner (1964), and we correlate our results with the behaviour of the coupling tensor. In particular, we show that falling motion without spin occurs in all directions if and only if the body is non-skew in the terminology of Brenner. We give a weakened symmetry condition which assures that the body is non-skew.

In §5 we present an isoperimetric inequality between the capacity of a body and its average terminal settling speed. This inequality has the interesting property that equality is attained for all ellipsoids. A classical symmetrization inequality then gives another isoperimetric inequality between the average settling speed and the volume of the body, equality being attained for the sphere.

In §6 we establish that the terminal speed of an axially symmetric body falling in the axial direction is less than twice its terminal speed when it falls in a direction perpendicular to the axis. Thus the limit, two, of the ratio of these speeds must be approached from below. In §7 we prove that for any convex slender body the difference between two and the ratio of the terminal speed in the long direction to that in a perpendicular direction is bounded by an explicit function of the aspect ratio ϵ which approaches zero with ϵ. The proof consists of applying our monotonicity theorem in order to compare the settling speeds of the body with those of inscribed and circumscribed ellipsoids. The convexity can be replaced by various smoothness assumptions which ensure the possibility of inscribing and circumscribing suitable ellipsoids.

In the process we obtain a rigorous justification for the first term of the series for the drag coefficients obtained from slender-body theory by Burgers (1938), Tuck (1964, 1970), Taylor (1969), Cox (1970), Tillett (1970) and Batchelor (1970). The slender-body theory for potential flows can be justified if the boundary conditions are satisfied to within a certain order in ϵ by noting that a harmonic function satisfies a maximum principle (see, e.g. Moran 1963, p. 295). No such principle is known for solutions of the Stokes equations.

Tillett has used the rather sophisticated ideas of inner and outer expansions of singular perturbation theory, but his argument leans heavily on the hypothesis that for sufficiently small ϵ his dual integral equations have an inverse which is uniformly bounded in the maximum norm. The very existence of such an inverse implies that the Stokes flow can always be extended inside the body to a Stokes flow which is singular only on the axis. Because of a scarcity of explicit solutions for bodies of the form $r \leqslant \epsilon R(z)$, we have not been able to find a counter example, but it seems unlikely that this hypothesis is always true.

Batchelor (1970) has extended the ideas of slender-body theory to obtain a formal approximation to the drag of an arbitrary slender body without an attempt at a rigorous proof. The following example shows that great care must be taken in defining the class of slender bodies to which the expansions are to apply. Consider a dumb-bell consisting of two spheres of radius ϵ connected by a cylindrical rod of length l and diameter $e^{-\epsilon^{-2}}$. For small ϵ this body is certainly slender, but we shall show in §7 that it behaves like two distant spheres. In particular, the ratio of the terminal speeds in the axial and perpendicular directions approaches one rather than two. Thus while the expansions of slender-body theory may well be correct, their correctness has not yet been proved. A rigorous proof of the correctness of the first term is therefore not redundant.

The interest in a solution of the Stokes equations lies in the fact that it is supposed to be an approximation to a solution of the Navier–Stokes equations. It will be shown elsewhere (Weinberger 1972) that the steady falling motions of a body in Stokes flow obtained here are limits as the dimensionless parameter $\rho m|\mathbf{g}|/\mu^2$ approaches zero of analogous steady falling motions of the same body in a Navier–Stokes fluid. Here ρ is the density of the fluid, μ is its viscosity and $m|\mathbf{g}|$ is the gravitational force minus the buoyant force acting on the body.

2. Formulation of the problem

Let D be the complement of a closed connected bounded set B with boundary $\dot{B}$ We consider the problem of the settling of the rigid body B under its own weight in a very viscous fluid which fills D. In the Stokes approximation the instantaneous velocity $\mathbf{u}$ and pressure p are solutions of the equations

$$\left.\begin{aligned} \sigma_{ij} &= \mu(u_{i,j}+u_{j,i}-p\delta_{ij}),\\ \sigma_{ij,j} &= 0,\\ u_{i,i} &= 0 \quad \text{in } D. \end{aligned}\right\} \qquad (2.1)$$

We shall use the usual summation convention and the subscript 'j' for $\partial/\partial x_j$. The constant μ is the viscosity; μp is the sum of the viscous pressure and the gravitational potential. We shall take account of the buoyant force due to the latter in setting up our equilibrium conditions.

Since B is rigid, the velocity must be of the form $\mathbf{U}+\boldsymbol{\omega}\times\mathbf{r}$ on the boundary $\dot{B}$. The vectors $\mathbf{U}$ and $\boldsymbol{\omega}$ are to be determined from the fact that the gravitational forces are in equilibrium with the viscous forces. These considerations lead to the boundary conditions

$$\left.\begin{aligned} &\mathbf{u} = \mathbf{U}+\boldsymbol{\omega}\times\mathbf{r} \quad \text{on} \quad \dot{B}, \\ &\mathbf{u}\to 0 \quad \text{at} \quad \infty, \\ &\oint_{\dot{B}} \mathbf{f}\,dS = m\mathbf{g}, \quad \oint_{\dot{B}} \mathbf{f}\times\mathbf{r}\,dS = m\mathbf{g}\times\mathbf{r}^{(c)}, \end{aligned}\right\} \tag{2.2}$$

where
$$f_i = \sigma_{ij}n_j.$$

Here $\mathbf{n}$ is the unit inward normal on $\dot{B}$ and $\mathbf{g}$ is the acceleration due to gravity. The mass m is the net mass, which is the difference between the mass m' of the body and the mass m'' of the displaced fluid. Similarly, $\mathbf{r}^{(c)}$ is the effective centre of mass, defined by

$$m\mathbf{r}^{(c)} = m'\mathbf{r}'-m''\mathbf{r}'',$$

where $\mathbf{r}'$ is the centre of mass of B and $\mathbf{r}''$ is the centroid of B.

Since all acceleration terms are neglected, equations (2.1) are valid in a moving rectangular co-ordinate system. In particular we choose a co-ordinate system attached to B. We observe that the rigid motion $\mathbf{u} = \mathbf{U}+\mathbf{r}\times\boldsymbol{\omega}$ satisfies the homogeneous equations (2.1) and gives zero stress. Accordingly if we replace the velocity $\mathbf{v}$ in our moving co-ordinates, which obviously vanishes on $\dot{B}$, by $\mathbf{u} = \mathbf{v}+\mathbf{U}+\boldsymbol{\omega}\times\mathbf{r}$, we find that this function again satisfies the problem (2.1), (2.2). We shall be interested in the situation in which the fall of the body B is steady, in the sense that $\mathbf{u}$ is independent of time in our moving co-ordinate system.

The downward vector $\mathbf{g}$ moves in the co-ordinate system attached to the body according to the law

$$d\mathbf{g}/dt = \boldsymbol{\omega}\times\mathbf{g}.$$

We see from the third equation in (2.2) that if $\mathbf{u}$ is independent of time so is $\mathbf{g}$. Hence we must have $\boldsymbol{\omega} = \lambda\mathbf{g}$ for some constant scalar λ. Conversely, if $\boldsymbol{\omega} = \lambda\mathbf{g}$ with λ constant, the conditions (2.2) are independent of time, so that any solution will be stationary.

We can now consider two possible problems of steady fall.

PROBLEM 1. Given B and the vector $\mathbf{g}$, find a position of the effective centre of mass $\mathbf{r}^{(c)}$, a vector $\mathbf{U}$ and a scalar λ such that equations (2.2) are satisfied with $\boldsymbol{\omega} = \lambda\mathbf{g}$.

PROBLEM 2. Given B and its mass distribution, find vectors $\mathbf{g}$ and $\mathbf{U}$ and a scalar λ such that the boundary conditions (2.2) are satisfied with $\boldsymbol{\omega} = \lambda\mathbf{g}$.

In problem 1 we think of B as a hollow body in which masses may be moved about to produce a motion in which a given direction $\mathbf{g}$ in B points downward. We shall mainly treat problem 1; however, we shall show (theorem 2) that problem 2 is a special case of problem 1.

In order to solve problem 1 we note that the torque equation in (2.2) can be solved for $\mathbf{r}^{(c)}$ if and only if the scalar product of the torque due to the viscous forces with $\mathbf{g}$ is zero. Thus, for problem 1 the boundary conditions reduce to

$$\left.\begin{aligned} &\mathbf{u} = \mathbf{U} + \lambda \mathbf{g} \times \mathbf{r} \quad \text{on} \quad \dot{B}, \\ &\mathbf{u} \to 0 \quad \text{at} \quad \infty, \\ &\oint_{\dot{B}} \mathbf{f}\, dS = m\mathbf{g}, \quad \mathbf{g} \cdot \oint_{\dot{B}} \mathbf{f} \times \mathbf{r}\, dS = 0, \end{aligned}\right\} \tag{2.3}$$

where $$f_i = \sigma_{ij} n_j.$$

If the problem (2.1), (2.3) has a solution, it is necessary and sufficient to put the effective centre of mass $\mathbf{r}^{(c)}$ on the vertical line

$$\mathbf{r}^{(c)} = \frac{-1}{m|\mathbf{g}|^2} \mathbf{g} \times \oint_{\dot{B}} \mathbf{f} \times \mathbf{r}\, dS + \alpha \mathbf{g}, \tag{2.4}$$

where α is an arbitrary constant, in order to realize the steady falling motion. Since $\mathbf{f}$ is easily seen to be proportional to the weight $m|\mathbf{g}|$, the line (2.4) depends only on the direction of $\mathbf{g}$ and is independent of m or $|\mathbf{g}|$. The existence of the solution of the problem (2.1), (2.3) will be proved in §3.

We define the settling speed of B to be the vertical component of $\mathbf{U}$, which is also the vertical speed of every point in B, and write

$$s(m, \mathbf{g}) = \mathbf{U} \cdot \mathbf{g}/|\mathbf{g}|. \tag{2.5}$$

Since $\mathbf{U}$ is proportional to the ratio $m|\mathbf{g}|/\mu$ we see that

$$s(m, \mathbf{g}) = (m/\mu)|\mathbf{g}|\, t(\mathbf{g}), \tag{2.6}$$

where the quantity $$t(\mathbf{g}) \equiv \mu \mathbf{U} \cdot \mathbf{g}/m|\mathbf{g}|^2 \tag{2.7}$$

depends only on the direction of $\mathbf{g}$ and the geometry of B. Moreover, it is easily seen that

$$t(-\mathbf{g}) = t(\mathbf{g}). \tag{2.8}$$

We shall call $t(\mathbf{g})$ the terminal settling speed; it is simply the settling speed in a system of units in which $m|\mathbf{g}|/\mu = 1$. Once $t(\mathbf{g})$ is known, $s(m, \mathbf{g})$ can immediately be computed from the definition (2.6).

Several of our results will be stated in terms of the capacity C of B which is defined as

$$C = \frac{1}{4\pi} \inf \int_D \phi_{,i} \phi_{,i}\, dx \tag{2.9}$$

over smooth functions ϕ which are unity in a neighbourhood of B and vanish at infinity. The minimum of the right-hand side is attained for a unique function h in the Sobolev space $H^{1,2}$. This function is the strong solution of the boundary-value problem

$$\left.\begin{aligned} &\Delta h = 0 \quad \text{in} \quad D, \\ &h = \begin{cases} 1 & \text{on} \quad \dot{B}, \\ 0 & \text{at} \quad \infty. \end{cases} \end{aligned}\right\} \tag{2.10}$$

Moreover, $$C = \frac{1}{4\pi} \oint_{\dot{B}} \frac{\partial h}{\partial n}\, dS. \tag{2.11}$$

Since (2.10) is a scalar problem and since harmonic functions satisfy a maximum principle it is much easier to approximate C than $t(\mathbf{g})$.

3. Variational inequalities for the terminal speed

For any two symmetric tensor fields τ_{ij} and ρ_{ij} which are square-integrable in D we define the symmetric bilinear form

$$E(\tau,\rho) = \frac{1}{2\mu}\int_D (\tau_{ij}-\tfrac{1}{3}\tau_{kk}\delta_{ij})(\rho_{ij}-\tfrac{1}{3}\rho_{ll}\delta_{ij})\,dx. \tag{3.1}$$

Let τ_{ij} be any smooth square-integrable tensor field which has the properties

$$\tau_{ij,j} = 0 \quad \text{in} \quad D, \quad g_i \oint_{\dot B} \epsilon_{ijk} x_j \tau_{kl} n_l dS = 0. \tag{3.2}$$

(Here ϵ_{ijk} is the usual alternating tensor.) Let ρ be of the form

$$\rho_{ij} = \mu(v_{i,j}+v_{j,i}-\Pi\delta_{ij}), \tag{3.3}$$

where $\mathbf{v}$ is any smooth vector field with compact support which satisfies the conditions

$$\left.\begin{aligned} &\operatorname{div}\mathbf{v} = 0,\\ &\mathbf{v} = \mathbf{V}+\gamma\mathbf{g}\times\mathbf{r} \quad \text{on} \quad \dot B,\\ &\mathbf{v}\to 0 \quad \text{at infinity}\end{aligned}\right\} \tag{3.4}$$

for some constant vector $\mathbf{V}$ and some scalar γ. We note that the tensor $\rho_{ij}-\frac{1}{3}\rho_{kk}\delta_{ij}$ is independent of the scalar function Π, so that the properties of Π are unimportant.

An integration by parts now shows that

$$\begin{aligned} E(\tau,\rho) &= \tfrac{1}{2}\int_D (\tau_{ij}-\tfrac{1}{3}\tau_{kk}\delta_{ij})(v_{i,j}+v_{j,i})\,dx\\ &= \int_D \tau_{ij}v_{i,j}\,dx = \oint_{\dot B} v_i\tau_{ij}n_j\,dS\\ &= V_i\oint_{\dot B}\tau_{ij}n_j\,dS + \gamma g_i\epsilon_{ijk}\int_{\dot B} x_j\tau_{jk}n_k\,dS.\end{aligned}$$

Therefore by (3.2)

$$E(\tau,\rho) = V_i\oint_{\dot B}\tau_{ij}n_j\,dS. \tag{3.5}$$

We have derived this equation on the assumption that τ and ρ are smooth. However, both sides of (3.5) are continuous in the L_2 norm

$$\left\{\int_D \tau_{ij}\tau_{ij}\,dx\right\}^{\frac{1}{2}}$$

provided that we interpret the right-hand side of (3.5) as

$$V_i\oint_{\dot B}\tau_{ij}n_j\,dS \equiv \int_D \tau_{ij}\phi_{i,j}\,dS,$$

where $\boldsymbol{\phi}$ is any C_∞ vector field which vanishes outside a bounded set and equals $\mathbf{V}$ in a neighbourhood of B. Therefore the identity (3.5) holds for τ_{ij} on the closure T_1 in L_2 of tensors which satisfy (3.2) and for an arbitrary ρ_{ij} on the closure T_2 in L_2 of tensor fields of the form (3.3) with $\mathbf{v}$ subject to the conditions (3.4). If the boundary is not smooth, we replace the boundary conditions in (3.4) by $\mathbf{v} = \mathbf{V}+\gamma\mathbf{g}\times\mathbf{r}$ in a neighbourhood of B and $\mathbf{v} = 0$ outside some bounded set in deriving (3.5) and in defining the closure T_2. The tensor fields in T_1 and in T_2 satisfy (3.2) and (3.3), respectively, in a weak sense.

If we assume for the moment that the boundary-value problem (2.1), (2.3) has a solution for which the stress σ_{ij} lies in $T_1 \cap T_2$ we see from (3.5) that

$$E(\sigma,\sigma) = m\mathbf{U}.\mathbf{g} = m|\mathbf{g}|s(m,\mathbf{g}) = (m^2|\mathbf{g}|^2/\mu)\,t(\mathbf{g}), \tag{3.6}$$

which is clearly the rate of dissipation of energy in the fluid owing to the viscosity.

We now choose $\tau = \sigma$ in (3.5) and let ρ be of the form (3.3), where $\mathbf{v}$ is any vector field which satisfies the conditions (3.4). We then see that

$$E(\sigma,\rho) = m\mathbf{V}.\mathbf{g}. \tag{3.7}$$

Since $\mathbf{v}$ is an arbitrarily chosen vector field the right-hand side of (3.7) as well as $E(\rho,\rho)$ can be evaluated.

Since the quadratic form $E(\tau,\tau)$ is positive semi-definite, Schwarz's inequality

$$E(\sigma,\rho)^2 \leqslant E(\sigma,\sigma)E(\rho,\rho)$$

is valid. From this and (3.7) we see that

$$E(\sigma,\sigma) \geqslant (m\mathbf{g}.\mathbf{V})^2/E(\rho,\rho). \tag{3.8}$$

For the purpose of simplifying the computations we observe that

$$\int_D (v_{i,j}+v_{j,i})(v_{k,j}+v_{j,i})\,dx = 2\int_D |\mathrm{curl}\, v|^2\,dx + 4\int_D v_{i,j}v_{j,i}\,dx.$$

Since $\mathrm{div}\,\mathbf{v} = 0$

$$\int_D v_{i,j}v_{j,i}\,dx = \oint_{\dot{B}} v_{i,j}v_j n_i\,dS = \oint_{\dot{B}} v_j(v_{i,j}n_i - v_{i,i}n_j)\,dS.$$

Because the last integral involves only $\mathbf{v}$ and its tangential derivatives on $\dot{B}$, we may replace $\mathbf{v}$ in the integral by its boundary values $\mathbf{V}+\gamma\mathbf{g}\times\mathbf{r}$. In this way we find that

$$\int_D v_{i,j}v_{j,i}\,dx = 2|B||\gamma\mathbf{g}|^2, \tag{3.9}$$

where $|B|$ is the volume of B. Therefore

$$E(\rho,\rho) = \mu\left\{\int_D |\mathrm{curl}\,\mathbf{v}|^2\,dx + 4|B||\gamma\mathbf{g}|^2\right\} \tag{3.10}$$

and the inequality (3.8) may be written in the form

$$E(\sigma,\sigma) \geqslant \frac{(m\mathbf{V}.\mathbf{g})^2}{\mu\left\{\int_D |\mathrm{curl}\,\mathbf{v}|^2\,dx + 4|B||\gamma\mathbf{g}|^2\right\}}. \tag{3.11}$$

We again carry through this derivation for smooth $\mathbf{v}$, and then extend it to ρ in T_2. We have assumed, however, that the problem (2.1), (2.3) has a solution σ in

$T_1 \cap T_2$. It follows from the identity (3.9) that the denominator $E(\rho,\rho)$ of the right-hand side of (3.11) satisfies the inequality

$$E(\rho,\rho) \geqslant \mu \int_D v_{i,j} v_{i,j} dx. \tag{3.12}$$

It is easily seen that because of the conditions (3.4) the quadratic form in the numerator on the right of (3.11) is compact with respect to this square integral. It follows that, if the supremum of the right-hand side of (3.11) is finite, it is attained for some tensor field σ on the closed space T_2. The usual Euler equation argument then shows that if σ is properly normalized it also lies in T_1 and satisfies all conditions of the problem (2.1), (2.3). Moreover, this solution is easily seen to be unique.

We now note that by (3.12) and the definition (2.9)

$$E(\rho,\rho) \geqslant \mu \int_D v_{i,j} v_{i,j} dx \geqslant \frac{\mu}{|\mathbf{g}|^2} \int_D |\mathrm{grad}\,(\mathbf{v}.\mathbf{g})|^2 dx$$
$$\geqslant \mu \frac{(\mathbf{V}.\mathbf{g})^2}{|\mathbf{g}|^2} \int_D \left|\mathrm{grad}\left(\frac{\mathbf{v}.\mathbf{g}}{\mathbf{V}.\mathbf{g}}\right)\right|^2 dx \geqslant 4\pi\mu C(\mathbf{V}.\mathbf{g})^2/|\mathbf{g}|^2.$$

Thus the right-hand side of (3.11) is bounded by $|m\mathbf{g}|^2/4\pi\mu C$. We conclude that the supremum is finite whenever the capacity of B is positive. Finally, we observe that T_2 contains all continuous piecewise continuously differentiable vector fields which satisfy the conditions (3.4). We thus have proved the following result.

THEOREM 1. If B is any body with positive capacity, there exists a unique weak solution of the problem (2.1), (2.3). The solution $\mathbf{u}$ gives the maximum of the right-hand side of (3.11) over all continuous, piecewise continuously differentiable, solenoidal vector fields with square-integrable gradient which vanish at infinity and which are of the form $\mathbf{V}+\gamma\mathbf{g}\times\mathbf{r}$ on $\dot{B}$.

Moreover, the terminal settling speed $t(\mathbf{g})$ satisfies the inequality

$$t(\mathbf{g}) \geqslant 1/4\pi C \tag{3.13}$$

for every direction of fall.

We observe that the constant 4π in the inequality (3.13) is sharp in the sense that for a prolate spheroid of major axis l and minor axis ϵl the product $4\pi C t(\mathbf{g})$ approaches one as $\epsilon \to 0$, provided $\mathbf{g}$ lies along the major axis. Note that since $\mathbf{g}$ occurs in the admissibility condition (3.4) as well as in the numerator of (3.11), $t(\mathbf{g})$ is not a quadratic functional of $\mathbf{g}$ (see formula (4.13) below).

We shall now show that for any body with a fixed mass distribution there is at least one direction in which the body has a steady settling motion.

THEOREM 2. For any closed bounded connected body B with positive capacity and with given net mass m and effective centre of mass $\mathbf{r}^{(c)}$ there is a direction $\mathbf{g}$ such that the problem (2.1), (2.2) is satisfied with $\boldsymbol{\omega} = \lambda\mathbf{g}$.

Proof. Choose the origin at the effective centre of mass, so that $\mathbf{r}^{(c)} = 0$. Let σ be the stress in the solution of the problem (2.1), (2.3) and let $\mathbf{f}$ be the corresponding surface force. Then for each vector $\mathbf{g}$, the torque

$$\mathbf{L}(\mathbf{g}) \equiv \oint_{\dot{B}} \mathbf{f}\times\mathbf{r}\, dS$$

is a vector perpendicular to $\mathbf{g}$. It is clear from theorem 1 that this vector depends continuously on $\mathbf{g}$. Thus $\mathbf{L}(\mathbf{g})$ represents a tangential field on the sphere

$$|\mathbf{g}| = \text{constant}.$$

By a well-known theorem of topology (see, e.g. Chinn & Steenrod 1966, theorem 34.1) there must be a direction $\mathbf{g}$ for which $\mathbf{L}(\mathbf{g}) = 0$. For this $\mathbf{g}$ the conditions (2.1) and (2.2) are satisfied with $\boldsymbol{\omega} = \lambda\mathbf{g}$. (Note that $-\mathbf{g}$ also satisfies the condition.)

Another result of the same kind is the following.

THEOREM 3. Given any direction $\mathbf{g}_0$, there exists a perpendicular direction $\mathbf{g}_1$ and a position $\mathbf{r}^{(c)}$ for the effective centre of mass such that the body B of positive capacity has steady falling motions in both the directions $\mathbf{g}_0$ and $\mathbf{g}_1$.

Proof. Solve the problem (2.1), (2.3) for $\mathbf{g} = \mathbf{g}_0$ and choose the origin to lie on the corresponding line (2.4). Then this line of possible centres of mass becomes $\mathbf{r}^{(c)} = \alpha\mathbf{g}_0$ and the torque equilibrium condition for a motion in another direction $\mathbf{g}$ becomes

$$\mu \oint_{\dot{B}} \mathbf{f} \times \mathbf{r}\, dS = \alpha m \mathbf{g} \times \mathbf{g}_0.$$

This equation can be solved for α if and only if the torque on the left is perpendicular to both $\mathbf{g}$ and $\mathbf{g}_0$. Thus, if $\mathbf{f}$ is the force which corresponds to the solution of the problem (2.1), (2.3) for $\mathbf{g}$, the condition becomes

$$\mathbf{g}_0 . \mathbf{L}(\mathbf{g}) \equiv \mathbf{g}_0 . \oint_{\dot{B}} \mathbf{f} \times \mathbf{r}\, dS = 0.$$

The left-hand side is an odd continuous function of $\mathbf{g}$. Therefore we see by letting $\mathbf{g}$ vary on the circle $\mathbf{g}.\mathbf{g}_0 = 0$, $|\mathbf{g}| = 1$ that there must be at least one $\mathbf{g}$, say $\mathbf{g} = \mathbf{g}_1$, for which the condition holds. For this $\mathbf{g}_1$, then, we can find an α such that, if $\mathbf{r}^{(c)} = \alpha\mathbf{g}_0$, there are steady flows in the $\mathbf{g}_0$ and $\mathbf{g}_1$ directions. (Note that we may replace $\mathbf{g}_0$ by $-\mathbf{g}_0$, $\mathbf{g}_1$ by $-\mathbf{g}_1$, or both.)

In order to obtain an upper bound for $t(\mathbf{g})$ we choose an arbitrary symmetric tensor field τ_{ij} which satisfies the conditions (3.2) and the additional condition

$$\oint_{\dot{B}} \tau_{ij} n_j\, dS = m g_i. \tag{3.14}$$

We apply the identity (3.5) to this τ and to $\rho = \sigma$, where σ is the stress in the solution of the problem (2.1), (2.3). Then (3.5) becomes

$$E(\tau, \sigma) = \mathbf{U}.m\mathbf{g} = E(\sigma, \sigma).$$

We again use Schwarz's inequality for the non-negative quadratic form $E(\rho, \rho)$ and find that

$$E(\sigma, \sigma)^2 = E(\tau, \sigma)^2 \leqslant E(\tau, \tau) E(\sigma, \sigma),$$

or

$$E(\sigma, \sigma) \leqslant E(\tau, \tau). \tag{3.15}$$

This inequality may be stated as a minimum principle.

THEOREM 4. The stress σ in the solution of the problem (2.1), (2.3) gives the minimum of the dissipation functional $E(\tau, \tau)$ over all symmetric stress fields which satisfy the conditions (3.2) and (3.14).

We remark that equality is attained in (3.15) if and only if τ differs from σ only by a hydrostatic stress. As a simple application of theorem 4 we shall prove the monotonicity of the settling speed as a functional of domain.

THEOREM 5. Let D and D' be two domains with bounded connected complements B and B', and suppose that B' contains B. Let B and B' have the same net mass m. Then the settling speeds $s(m, \mathbf{g})$ and $s'(m, \mathbf{g})$ in any direction $\mathbf{g}$ satisfy the inequality $s'(m, \mathbf{g}) \leqslant s(m, \mathbf{g})$.

Proof. Let σ be the stress which corresponds to the solution of the problem (2.1), (2.3) for D. It clearly satisfies the conditions (3.2) and (3.14). Since $D' \subset D$, σ is admissible in the minimum principle of theorem 4 for B'. Moreover, the dissipation functional $E'(\sigma, \sigma)$ integrates σ over a smaller set than D. Hence in obvious notation

$$E'(\sigma', \sigma') \leqslant E(\sigma, \sigma),$$

or

$$s'(m, \mathbf{g}) \leqslant s(m, \mathbf{g}).$$

(We could also write this result in the form $t'(\mathbf{g}) \leqslant t(\mathbf{g})$.)

Theorem 5 states that increasing the size of the body B decreases its terminal speed, provided that the net weight is not increased. That is, the weight of B' exceeds that of B by at most the weight of the additional fluid displaced by B'. The result is not in general true if B' has the same mass density as B. For example, the terminal speed of a sphere of fixed mass density is proportional to the square of its radius and hence increases with increasing size.

Note. As we pointed out in the proof of theorem 1, $m^2|\mathbf{g}|^2 t(\mathbf{g})/\mu$ is the supremum of the right-hand side of (3.11) over all $\mathbf{v}$ which satisfy (3.4) with $\mathbf{v}$ of the form

$$\mathbf{V} + \gamma \mathbf{g} \times \mathbf{r}$$

in some neighbourhood of B. Such a $\mathbf{v}$ is then admissible in the maximum principle for some B' which contains a neighbourhood of B. We see by theorem 5 that $t(\mathbf{g})$ is the upper limit of the speeds $t'(\mathbf{g})$ of the domains B' which contain a neighbourhood of B.

If we apply the identity (3.10) to the solution $\mathbf{u}$ of (2.1), (2.3) we find that

$$E(\sigma, \sigma) = \mu \left\{ \int_D |\operatorname{curl} \mathbf{u}|^2 \, dx + 4|B| \lambda^2 |\mathbf{g}|^2 \right\}. \tag{3.16}$$

Moreover, we see from the conditions (2.3) that

$$\begin{aligned} \oint_B \mathbf{n} . [(\operatorname{curl} \mathbf{u}) \times (\mathbf{g} \times \mathbf{r}) + p(\mathbf{g} \times \mathbf{r})] \, dS &= \oint_B [n_i(u_{i,j} - u_{j,i}) + p n_j] \, \epsilon_{jkl} g_k x_l \, dS \\ &= 2 \oint_B n_i u_{i,j} \epsilon_{jkl} g_k x_l \, dS = 2 \oint_B (n_i u_{i,j} - n_j . u_{i,i}) \, \epsilon_{jkl} g_k x_l \, dS. \end{aligned}$$

Since only tangential derivatives are involved in the integrand, we may replace $\mathbf{u}$ by its boundary value $\mathbf{U} + \lambda \mathbf{g} \times \mathbf{r}$. In this way we find that

$$\oint_B \mathbf{n} . [(\operatorname{curl} \mathbf{u}) \times (\mathbf{g} \times \mathbf{r}) + p \mathbf{g} \times \mathbf{r}] \, dS = 4|B||\mathbf{g}|^2 \lambda. \tag{3.17}$$

Thus if the volume $|B|$ is not zero, we can eliminate λ from (3.16) to obtain

$$E(\sigma,\sigma) = \mu\left\{\int_D |\operatorname{curl}\mathbf{u}|^2 dx + \frac{1}{4|B||\mathbf{g}|^2}\left(\oint_{\dot{B}} \mathbf{n}.[(\operatorname{curl}\mathbf{u})\times(\mathbf{g}\times\mathbf{r}) + p\mathbf{g}\times\mathbf{r}]\,dS\right)^2\right\}. \tag{3.18}$$

Now let $\{\mathbf{w}, \Pi\}$ be a pair consisting of a vector field and a scalar field, and let $\{\mathbf{q}, \phi\}$ be another such pair. We define the positive semi-definite bilinear form

$$\tilde{E}(\{\mathbf{w},\Pi\},\{\mathbf{q},\phi\}) \equiv \mu\left\{\int_D \mathbf{w}.\mathbf{q}\,dx + \frac{1}{4|B||\mathbf{g}|^2}\oint_{\dot{B}} \mathbf{n}.[\mathbf{w}\times(\mathbf{g}\times\mathbf{r}) + \Pi\mathbf{g}\times\mathbf{r}]\,dS \oint_{\dot{B}} \mathbf{n}.[\mathbf{q}\times(\mathbf{g}\times\mathbf{r}) + \phi\mathbf{g}\times\mathbf{r}]\,dS\right\}, \tag{3.19}$$

so that (3.18) becomes

$$\tilde{E}(\{\operatorname{curl}\mathbf{u}, p\},\{\operatorname{curl}\mathbf{u}, p\}) = E(\sigma,\sigma) = \frac{|m\mathbf{g}|^2}{\mu} t(\mathbf{g}).$$

We now choose an arbitrary pair $\{\mathbf{w}, \Pi\}$ which satisfies the conditions

$$\left.\begin{aligned} \operatorname{curl}\mathbf{w} + \operatorname{grad}\Pi &= 0,\\ \mu\oint_{\dot{B}}(\mathbf{w}\times\mathbf{n} - \Pi\mathbf{n})\,dS &= m\mathbf{g},\\ \mathbf{w}\to 0,\quad \Pi &\to 0 \text{ at } \infty. \end{aligned}\right\} \tag{3.20}$$

It is easily seen that

$$\begin{aligned} \int_D \mathbf{w}.\operatorname{curl}\mathbf{u}\,dx &= \oint_{\dot{B}} \mathbf{w}.\mathbf{n}\times\mathbf{u}\,dS + \int_D \mathbf{u}.\operatorname{curl}\mathbf{w}\,dx\\ &= \oint_{\dot{B}}(\mathbf{w}.\mathbf{n}\times\mathbf{u} - \Pi\mathbf{u}.\mathbf{n})\,dS\\ &= \frac{1}{\mu}\mathbf{U}.m\mathbf{g} - \lambda\oint_{\dot{B}} \mathbf{n}.[\mathbf{w}\times(\mathbf{g}\times\mathbf{r}) + \Pi\mathbf{g}\times\mathbf{r}]\,dS. \end{aligned}$$

Therefore we see from the identity (3.17) that

$$\tilde{E}(\{\mathbf{w},\Pi\},\{\mathbf{u},p\}) = \mathbf{U}.m\mathbf{g} = \tilde{E}(\{\mathbf{u},p\},\{\mathbf{u},p\}).$$

By the reasoning that was used to derive the inequality (3.15) we see that

$$\tilde{E}(\{\mathbf{u},p\},\{\mathbf{u},p\}) \leqslant \tilde{E}(\{\mathbf{w},\Pi\},(\mathbf{w},\Pi\}). \tag{3.21}$$

We note that if B' is any bounded set containing B, then by the divergence theorem

$$\left.\begin{aligned} \oint_{\dot{B}}(\mathbf{w}\times\mathbf{n} - \Pi\mathbf{n})\,dS &= \oint_{\dot{B}'}(\mathbf{w}\times\mathbf{n} - \Pi\mathbf{n})\,dS,\\ \oint_{\dot{B}} \mathbf{n}.[\mathbf{w}\times(\mathbf{g}\times\mathbf{r}) + \Pi\mathbf{g}\times\mathbf{r}]\,dS &\\ = \oint_{\dot{B}'} \mathbf{n}.[\mathbf{w}\times(\mathbf{g}\times\mathbf{r}) &+ \Pi\mathbf{g}\times\mathbf{r}]\,dS - 2\int_{D-D'} \mathbf{g}.\mathbf{w}\,dx. \end{aligned}\right\} \tag{3.22}$$

If $\dot{B}$ is not a smooth surface, we can find a smooth surface $\dot{B}'$ in any neighbourhood of B and define the boundary integrals which occur in (3.19) and (3.20) by means of these identities.

If the volume $|B|$ of B is zero, we apply the inequality (3.21) to any B' with positive volume $|B'|$ which contains a neighbourhood of B. We see by (3.22) that a pair $\{\mathbf{w}, \Pi\}$ which satisfies the conditions (3.20) for B also satisfies them for B' and we see by (3.21) that

$$\frac{|m\mathbf{g}|^2}{\mu} t'(\mathbf{g}) \leqslant \mu\left\{\int_{D'} |\mathbf{w}|^2 dx + \frac{1}{4|B'|\,|\mathbf{g}|^2}\left[\oint_{\dot{B}'} \mathbf{n}.[\mathbf{w}\times(\mathbf{g}\times\mathbf{r}) + \Pi\mathbf{g}\times\mathbf{r}]\,dS\right]^2\right\},$$

where $t'(\mathbf{g})$ is the terminal settling speed of B'.

Now suppose that

$$\oint_{\dot{B}} \mathbf{n}.[\mathbf{w}\times(\mathbf{g}\times\mathbf{r}) + \Pi\mathbf{g}\times\mathbf{r}]\,dS = 0, \tag{3.23}$$

in the sense of the second identity in (3.22). It then follows from this identity and Schwarz's inequality that

$$\begin{aligned} \frac{|m\mathbf{g}|^2}{\mu} t'(\mathbf{g}) &\leqslant \mu\left\{\int_{D'} |\mathbf{w}|^2 dx + \frac{1}{|\mathbf{g}|^2}\int_{D-D'} (\mathbf{w}.\mathbf{g})^2 dx\right\} \\ &\leqslant \mu \int_D |\mathbf{w}|^2 dx \end{aligned} \tag{3.24}$$

for any domain D' whose closure lies in D and whose complement B' is bounded. It follows by the note after theorem 5 that

$$\frac{|m\mathbf{g}|^2}{\mu} t(\mathbf{g}) = E(\sigma, \sigma) \leqslant \mu \int_D |\mathbf{w}|^2 dx.$$

Finally, we observe that it is easily verified that the pair $\{\operatorname{curl} \mathbf{u}, p\}$ satisfies the conditions (3.20) and, if $|B| = 0$, the condition (3.23). Thus we can again summarize our results as a minimum principle.

We define the pair $\{\mathbf{w}, \Pi\}$ to be admissible if it satisfies the conditions (3.20) and, if $|B| = 0$, also satisfies (3.23). We define

$$\tilde{E}(\{\mathbf{w}, \Pi\}, \{\mathbf{w}, \Pi\}) = \begin{cases} \mu\left\{\int_D |\mathbf{w}|^2 dx + \frac{1}{4|B|\,|\mathbf{g}|^2}\left[\oint_{\dot{B}} \mathbf{n}.[\mathbf{w}\times(\mathbf{g}\times\mathbf{r}) + \Pi\mathbf{g}\times\mathbf{r}]\,dS\right]^2\right\}, & |B| \neq 0, \\ \mu \int_D |\mathbf{w}|^2 dx, & |B| = 0. \end{cases}$$

THEOREM 6. The pair $\{\operatorname{curl} \mathbf{u}, p\}$, where $\mathbf{u}$ and p solve the boundary-value problem (2.1), (2.3), gives the minimum of the quadratic form $\tilde{E}(\{\mathbf{w}, \Pi\}, \{\mathbf{w}, \Pi\})$ over all admissible pairs $\{\mathbf{w}, \Pi\}$. Moreover, the minimum value is equal to $|m\mathbf{g}|^2 t(\mathbf{g})/\mu$.

We note that although theorems 4 and 6 both provide upper bounds for the energy dissipation, one involves a tensor and the other a pair $\{\mathbf{w}, \Pi\}$. Moreover, while the tensor τ must satisfy three differential equations and two integral conditions in (3.2) and (3.14), the pair $\{\mathbf{w}, \Pi\}$ is subjected to three differential equations and a single integral condition in (3.20). Thus, it is easier to find admissible pairs for theorem 6 than to find admissible tensors τ for theorem 4.

As we shall see, theorem 7 (see §5) follows from theorem 6, but we have not been able to derive it from theorem 4.

4. Stability of steady fall

We wish to justify the name 'terminal settling speed' by showing that at least in some circumstances the quasi-steady falling motion of B converges to a steady one. Brenner (1964) (see also Happel & Brenner 1965, p. 173) has shown that the viscous force $\mathbf{F}$ and torque $\mathbf{L}$ due to the velocity satisfying (2.1) with the given boundary velocity

$$\mathbf{u} = \mathbf{U} + \boldsymbol{\omega} \times \mathbf{r} \quad \text{on} \quad \dot{B}$$

can be expressed in the form

$$\mathbf{F} = \mu(\mathbf{K}\mathbf{U} + \mathbf{C}^*\boldsymbol{\omega}), \quad \mathbf{L} = \mu(\mathbf{C}\mathbf{U} + \boldsymbol{\Omega}\boldsymbol{\omega}), \tag{4.1}$$

where

$$\begin{pmatrix} \mathbf{K} & \mathbf{C}^* \\ \mathbf{C} & \boldsymbol{\Omega} \end{pmatrix} \tag{4.2}$$

is a symmetric positive-definite 6×6 matrix. (Although Brenner only derived his result for smooth bodies, it can be extended to any bounded connected body with positive capacity by the techniques used in proving theorem 1.) The matrix (4.2) can be determined experimentally by measuring the force and torque when the body B is rigidly suspended and subjected to slow translational and rotational flows in each of three orientations.

The equations of motion for the quasi-steady fall of B can therefore be written in the form

$$\left.\begin{aligned} \mu(\mathbf{K}\mathbf{U} + \mathbf{C}^*\boldsymbol{\omega}) &= m\mathbf{g}, \\ \mu(\mathbf{C}\mathbf{U} + \boldsymbol{\Omega}\boldsymbol{\omega}) &= m\mathbf{g} \times \mathbf{r}^{(c)}, \\ d\mathbf{g}/dt &= \boldsymbol{\omega} \times \mathbf{g}. \end{aligned}\right\} \tag{4.3}$$

We solve the first two equations for $\boldsymbol{\omega}$ and substitute in the third to obtain the equation

$$\mu \, d\mathbf{g}/dt = -m\mathbf{g} \times [\boldsymbol{\Omega} - \mathbf{C}\mathbf{K}^{-1}\mathbf{C}^*]^{-1} [\mathbf{g} \times \mathbf{r}^{(c)} - \mathbf{C}\mathbf{K}^{-1}\mathbf{g}]. \tag{4.4}$$

We now take a particular downward direction $\mathbf{g}_0$ and choose the origin so that it lies on the line (2.4) when $\mathbf{g} = \mathbf{g}_0$. Then there is a steady fall in the direction $\mathbf{g}_0$ provided that

$$\mathbf{r}^{(c)} = \alpha \mathbf{g}_0. \tag{4.5}$$

We wish to show that α can be chosen to make this motion stable.

By hypothesis, when $\mathbf{g} = \mathbf{g}_0$ and $\mathbf{r}^{(c)} = \alpha\mathbf{g}_0$ the corresponding $\boldsymbol{\omega}$ is $\lambda\mathbf{g}_0$. Hence we see from the first two equations of (4.3) that

$$-m/\mu[\boldsymbol{\Omega} - \mathbf{C}\mathbf{K}^{-1}\mathbf{C}^*]^{-1} \mathbf{C}\mathbf{K}^{-1}\mathbf{g}_0 = \lambda\mathbf{g}_0. \tag{4.6}$$

(In fact, this is a necessary and sufficient condition for the existence of a steady flow in the direction $\mathbf{g}_0$ when $\mathbf{r}^{(c)} = 0$. Thus, theorem 2 follows from the fact that a real 3×3 matrix has at least one real eigenvalue.)†

We see from the last equation in (4.3) that $|\mathbf{g}|^2$ is constant in time. Thus, $|\mathbf{g}|^2 = |\mathbf{g}_0|^2$ and hence

$$|\mathbf{g} - \mathbf{g}_0|^2 = 2(|\mathbf{g}_0|^2 - \mathbf{g} \cdot \mathbf{g}_0).$$

† H. Brenner has been kind enough to send me an unpublished dissertation of his student A. V. Goldman (1966). In §§5 and 6 of volume I, this dissertation presents a formulation of problem 2 which is equivalent to (4.6), as well as some stability considerations.

Therefore

$$\frac{\mu}{2m}\frac{d}{dt}|\mathbf{g}-\mathbf{g}_0|^2 = \mathbf{g}_0 . \mathbf{g}\times[\mathbf{\Omega}-\mathbf{CK}^{-1}\mathbf{C}^*]^{-1}[\alpha\mathbf{g}\times\mathbf{g}_0-\mathbf{CK}^{-1}\mathbf{g}]$$
$$= -(\mathbf{g}\times\mathbf{g}_0).[\mathbf{\Omega}-\mathbf{CK}^{-1}\mathbf{C}^*]^{-1}\left[\alpha\mathbf{g}\times\mathbf{g}_0-\mathbf{CK}^{-1}\left(\mathbf{g}-\frac{\mathbf{g}.\mathbf{g}_0}{|\mathbf{g}_0|^2}\mathbf{g}_0\right)\right] \tag{4.7}$$

by (4.4)–(4.6). Because the 6×6 matrix (4.2) is positive-definite, the 3×3 matrix $\mathbf{\Omega}-\mathbf{CK}^{-1}\mathbf{C}^*$ is also positive-definite. Therefore there is a positive constant γ such that

$$(\mathbf{g}\times\mathbf{g}_0).[\mathbf{\Omega}-\mathbf{CK}^{-1}\mathbf{C}^*]^{-1}(\mathbf{g}\times\mathbf{g}_0) \geqslant \gamma|\mathbf{g}\times\mathbf{g}_0|^2. \tag{4.8}$$

Also, since

$$\mathbf{g}-\frac{\mathbf{g}.\mathbf{g}_0}{|\mathbf{g}_0|^2} = \frac{1}{|\mathbf{g}_0|^2}\mathbf{g}_0\times(\mathbf{g}\times\mathbf{g}_0),$$

there is a constant δ such that

$$\left|(\mathbf{g}\times\mathbf{g}_0).[\mathbf{\Omega}-\mathbf{CK}^{-1}\mathbf{C}^*]^{-1}\mathbf{CK}^{-1}\left[\mathbf{g}-\frac{\mathbf{g}.\mathbf{g}_0}{|\mathbf{g}_0|^2}\right]\right| \leqslant \frac{\delta}{|\mathbf{g}_0|}\left|\mathbf{g}\times\mathbf{g}_0\right|^2. \tag{4.9}$$

Thus we find that

$$\frac{\mu}{2m}\frac{d}{dt}|\mathbf{g}-\mathbf{g}_0|^2 \leqslant -\left(\alpha\gamma-\frac{\delta}{|\mathbf{g}_0|}\right)|\mathbf{g}\times\mathbf{g}_0|^2. \tag{4.10}$$

Finally we observe that

$$|\mathbf{g}\times\mathbf{g}_0|^2 = |\mathbf{g}_0|^4-(\mathbf{g}.\mathbf{g}_0)^2 = |\mathbf{g}-\mathbf{g}_0|^2[|\mathbf{g}_0|^2-\tfrac{1}{4}|\mathbf{g}-\mathbf{g}_0|^2],$$

so that

$$\frac{d}{dt}|\mathbf{g}-\mathbf{g}_0|^2 \leqslant -\frac{2m}{\mu}\left(\alpha\gamma-\frac{\delta}{|\mathbf{g}_0|}\right)|\mathbf{g}-\mathbf{g}_0|^2[|\mathbf{g}_0|^2-\tfrac{1}{4}|\mathbf{g}-\mathbf{g}_0|^2].$$

It is then easily seen that if

$$\alpha|\mathbf{g}_0| > \delta/\gamma$$

$\mathbf{g}$ converges to $\mathbf{g}_0$ exponentially unless $\mathbf{g} = -\mathbf{g}_0$ initially. Thus, for sufficiently large α (that is, when the centre of mass is sufficiently low) the steady motion with $\mathbf{g} = \mathbf{g}_0$ is the limit of all falling motions except for the unstable motion with $\mathbf{g} = -\mathbf{g}_0$. Similarly, we see that for α sufficiently negative, the motion $\mathbf{g} = \mathbf{g}_0$ is unstable and the motion $\mathbf{g} = -\mathbf{g}_0$ is stable. It appears likely that for most bodies there is an intermediate region of α where neither motion is stable.

If the coupling matrix $\mathbf{C} = 0$, then $\delta = 0$ *and so the motion in the* $\mathbf{g}_0$ *direction* is stable for $\alpha < 0$, while the motion in the $-\mathbf{g}_0$ direction is stable for $\alpha > 0$. Following Brenner (1964), we call a body B a non-skew body if a proper choice of origin (the centre of stress) makes $\mathbf{C} = 0$.

In terms of the matrix (4.2) the steady-fall problem (2.1), (2.3) becomes

$$\left.\begin{aligned}\mu[\mathbf{KU}+\lambda\mathbf{C}^*\mathbf{g}] &= m\mathbf{g},\\ \mathbf{g}.[\mathbf{CU}+\lambda\mathbf{\Omega}\mathbf{g}] &= 0.\end{aligned}\right\} \tag{4.11}$$

It follows from these equations and the definition (2.7) that

$$\lambda = -\frac{m}{\mu}\frac{\mathbf{g}.\mathbf{CK}^{-1}\mathbf{g}}{\mathbf{g}.[\mathbf{\Omega}-\mathbf{CK}^{-1}\mathbf{C}^*]\mathbf{g}} \tag{4.12}$$

and

$$t(\mathbf{g}) = \frac{1}{|\mathbf{g}|^2}\left\{\mathbf{g}.\mathbf{K}^{-1}\mathbf{g}+\frac{(\mathbf{g}.\mathbf{CK}^{-1}\mathbf{g})^2}{\mathbf{g}.[\mathbf{\Omega}-\mathbf{CK}^{-1}\mathbf{C}^*]\mathbf{g}}\right\}. \tag{4.13}$$

These expressions are independent of the choice of the origin. Brenner (1964) (see also Happel & Brenner 1965, p. 174) has shown that for a suitable choice of the origin (the centre of reaction) $\mathbf{C}$ is symmetric. Since $\mathbf{K}$ is positive-definite and symmetric, $\mathbf{C}$ and $\mathbf{K}$ are then both diagonal in some (skew) co-ordinate system. In this way we see that a body B has a steady fall without spin ($\lambda = 0$) in any direction if and only if it is non-skew ($\mathbf{C} = 0$ when the origin is at the centre of reaction).

If $\mathbf{g}$ lies along a principal axis of $\mathbf{K}$, the drag coefficient evaluated by solving the problem (2.1) with $\mathbf{u} = \mathbf{g}$ on $\dot{B}$ is equal to $|\mathbf{g}|^2/(\mathbf{g}\,.\,\mathbf{K}^{-1}\mathbf{g})$. Thus we see from (4.13) that if the body is non-skew and $\mathbf{g}$ lies along a principal axis of $\mathbf{K}$, then $t(\mathbf{g})$ is the reciprocal of the drag coefficient so determined. If the body is not non-skew, $t(\mathbf{g})$ will, in general, be larger than this reciprocal. Brenner (1964) (see also, Happel & Brenner 1965, pp. 187–188) has shown that B is non-skew if it is either axially symmetric or orthotropic. We wish to show that the latter class can be extended.

Let B be symmetric about the origin in the sense that it is invariant under the transformation $(x_1, x_2, x_3) \to (-x_1, -x_2, -x_3)$. It then follows from the uniqueness of the solution of the problem (2.1), (2.3) that

$$\mathbf{u}(-x_1, -x_2, -x_3) = \mathbf{u}(x_1, x_2, x_3).$$

Since $\mathbf{g} \times \mathbf{r}$ is an odd function of $\mathbf{r}$, $\lambda = 0$ for any $\mathbf{g}$, so that B is non-skew. Thus any B which is symmetric about a point is non-skew. Clearly an orthotropic body (that is, a body which is symmetric about each of three orthogonal planes) is symmetric about a point.

5. An isoperimetric inequality between terminal speeds and capacity

We shall now prove the following result.

THEOREM 7. For any body B and any three mutually perpendicular directions $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ the inequality

$$t(\mathbf{i}) + t(\mathbf{j}) + t(\mathbf{k}) \leqslant 1/2\pi C \tag{5.1}$$

holds.

Proof. Let h be the capacitary potential defined in (2.10). For a given vector $\mathbf{g}$ we define the vector field

$$\mathbf{w} = (-m/4\pi C\mu)\, \mathbf{g} \times \operatorname{grad} h$$

and the scalar function

$$\Pi = (-m/4\pi C\mu)\, \mathbf{g}\,.\operatorname{grad} h.$$

This pair satisfies the equation

$$\operatorname{curl} \mathbf{w} + \operatorname{grad} \Pi = 0.$$

Moreover,

$$\mu \oint_{\dot{B}} [\mathbf{w} \times \mathbf{n} - \Pi \mathbf{n}]_i \, dS = -\frac{m}{4\pi C} \oint_{\dot{B}} \{(g_k h_{,i} - g_i h_{,k})\, n_k - g_k h_{,k} n_i\}\, dS$$
$$= \frac{m}{4\pi C} g_i \oint_{\dot{B}} \frac{\partial h}{\partial n} dS = m g_i.$$

(We have used the fact that $h = 1$ on $\dot{B}$, so that $\partial h/\partial x_i = n_i \partial h/\partial n$.) Thus the pair $\{\mathbf{w}, \Pi\}$ satisfies the conditions (3.20). In addition,

$$\oint_{\dot{B}} \mathbf{n}.[\mathbf{w}\times(\mathbf{g}\times\mathbf{r})+\Pi(\mathbf{g}\times\mathbf{r})]\,dS = -\frac{m}{4\pi C\mu}\oint_{\dot{B}}\frac{\partial h}{\partial n}[\mathbf{n}.\mathbf{g}\,\mathbf{r}.\mathbf{g}\times\mathbf{n}+\mathbf{g}.\mathbf{n}\mathbf{n}.\mathbf{g}\times\mathbf{r}]\,ds = 0,$$

so that the condition (3.23) is satisfied. Also,

$$\tilde{E}(\{\mathbf{w},\Pi\},\{\mathbf{w},\Pi\}) = \frac{m^2}{(4\pi C)^2\mu}\int_D |\mathbf{g}\times\operatorname{grad} h|^2\,dx. \tag{5.2}$$

We now observe that for any three mutually orthogonal unit vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$

$$|\mathbf{i}\times\operatorname{grad} h|^2+|\mathbf{j}\times\operatorname{grad} h|^2+|\mathbf{k}\times\operatorname{grad} h|^2 = 2|\operatorname{grad} h|^2. \tag{5.3}$$

We see from (5.2) and theorem 6 that

$$m^2 t(\mathbf{i}) \leqslant \frac{m^2}{(4\pi C)^2}\int_D |\mathbf{i}\times\operatorname{grad} h|^2\,dx$$

with similar inequalities for $\mathbf{j}$ and $\mathbf{k}$. If we add these three inequalities and use (5.3) and (2.9) we see that the inequality (5.1) is valid. Thus our theorem is proved.

It is interesting to observe that for any ellipsoid our function $\mathbf{w}$ is exactly the curl of the known velocity field. It follows that equality holds in (5.1) for all ellipsoids. Thus the inequality (5.1) is an isoperimetric one: of all bodies of given capacity the ellipsoids have the largest average terminal settling speed. Since equality holds in (5.1) for all ellipsoids, the right-hand side can be expected to be a good approximation to the left, at least for convex bodies.

Various inequalities are known for the capacity. The best-known one is the isoperimetric inequality of Poincaré, Faber and Szegö (see Polya & Szegö 1951)

$$C \geqslant \left(\frac{3|B|}{4\pi}\right)^{\frac{1}{3}},$$

where $|B|$ is the volume of B. If we combine this inequality with (5.1) we find that

$$t(i)+t(j)+t(k) \leqslant (6\pi^2|B|)^{-\frac{1}{3}}. \tag{5.4}$$

This is again an isoperimetric inequality: of all bodies of given volume the sphere has the largest average terminal settling speed.

We note that this average speed is for a given weight. If the net mass density ρ is prescribed, the average terminal speed $\frac{1}{3}[s(m,\mathbf{i})+s(m,\mathbf{j})+s(m,\mathbf{k})]$ is

$$\tfrac{1}{3}\rho|\mathbf{g}|^2|B|[t(\mathbf{i})+t(\mathbf{j})+t(\mathbf{k})]/\mu$$

and is therefore bounded by a constant times $|B|^{\frac{2}{3}}$. In particular, we see that for a body of fixed density the terminal speed approaches zero as the volume approaches zero.

6. A bound for the ratio of terminal speeds of an axially symmetric body

We consider an axially symmetric body B and orient the z axis along the axis of symmetry. When B is allowed to fall with its axis vertical, the resulting flow is clearly axially symmetric. It follows from the divergence condition that the velocity field can be represented in terms of a stream function† $\phi(r,z)$ by means of the relations

$$\mathbf{u} = \operatorname{curl}[(-y\mathbf{i}+x\mathbf{j})\,\phi], \tag{6.1}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the x and y directions, respectively. Since B is non-skew and the z direction lies along a principal axis, $\mathbf{u}$ must be equal to $t(\mathbf{k})\,\mathbf{k}$ on the boundary, provided that we make the weight $m|\mathbf{g}| = \mu$. It follows that

$$\left.\begin{aligned} &\phi = \tfrac{1}{2}t(k), \quad \operatorname{grad}\phi = 0 \quad \text{on} \quad \dot{B},\\ &\phi \to 0 \quad \text{at} \quad \infty. \end{aligned}\right\} \tag{6.2}$$

Moreover, ϕ is a solution of the differential equation

$$\Delta_5^2\phi = 0, \tag{6.3}$$

where

$$\Delta_5 \equiv \frac{\partial^2}{\partial r^2}+\frac{3}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2} \tag{6.4}$$

is the Laplace operator for an axially symmetric function in a five-dimensional space (see Weinstein 1953).

From (3.6) and (3.10) we see that

$$t(\mathbf{k}) = \iiint_D (\Delta_5\phi)^2 r^3\,dr\,dz\,d\theta. \tag{6.5}$$

We now construct a trial function for the flow corresponding to fall in the x direction. We let

$$\mathbf{v} = \operatorname{curl}(y\mathbf{k}\phi),$$

which satisfies the conditions (3.4) with $\mathbf{V} = \frac{1}{2}t(\mathbf{k})\,\mathbf{i}, \gamma = 0$. It follows from theorem 1 that

$$t(\mathbf{i}) \geqslant \frac{[\frac{1}{2}t(\mathbf{k})]^2}{\displaystyle\int_D |\operatorname{curl}\operatorname{curl}(y\mathbf{k}\phi)|^2\,dx}. \tag{6.6}$$

Now

$$\begin{aligned} \int_D |\operatorname{curl}\operatorname{curl}(y\mathbf{k}\phi)|^2\,dx &= \iiint\left|-\Delta(y\phi)\,\mathbf{k}+\operatorname{grad}\left(y\frac{\partial\phi}{\partial z}\right)\right|^2 r\,dr\,dz\,d\theta \\ &= \iiint\left\{\left[-\Delta_5\phi+\frac{\partial^2\phi}{\partial z^2}\right]^2 y^2+\left[\frac{\partial^2\phi}{\partial r\,\partial z}+\frac{1}{r}\frac{\partial\phi}{\partial z}\right]^2 y^2+\frac{1}{r^2}\left(\frac{\partial\phi}{\partial z}\right)^2 x^2\right\} r\,dr\,dz\,d\theta \\ &= \pi\iint\left\{\left[\frac{\partial^2\phi}{\partial r^2}+\frac{3}{r}\frac{\partial\phi}{\partial r}\right]^2 r^3+\left(\frac{\partial^2\phi}{\partial r\,\partial z}\right)^2 r^3+\frac{\partial}{\partial r}\left[r^2\left(\frac{\partial\phi}{\partial z}\right)^2\right]\right\} dr\,dz \\ &= \pi\iint\left\{\left[\frac{\partial^2\phi}{\partial r^2}+\frac{3}{r}\frac{\partial\phi}{\partial r}\right]^2+\left[\frac{\partial^2\phi}{\partial r\,\partial z}\right]^2\right\} r^3\,dr\,dz. \end{aligned} \tag{6.7}$$

† We have defined ϕ to be r^{-2} times the usual stream function.

Integration by parts shows that

$$\iiint (\Delta_5\phi)^2 r^3\,dr\,dz\,d\theta = 2\pi\iint\left\{\left[\frac{\partial^2\phi}{\partial r^2}+\frac{3}{r}\frac{\partial\phi}{\partial r}\right]^2+2\left[\frac{\partial^2\phi}{\partial r\,\partial z}\right]^2+\left[\frac{\partial^2\phi}{\partial z^2}\right]^2\right\} r^3\,dr\,dz.$$

Thus we see from (6.5) and (6.7) that

$$\int_D |\operatorname{curl}\operatorname{curl}(y\mathbf{k}\phi)|^2\,dx < \tfrac{1}{2}t(\mathbf{k}).$$

(Equality cannot hold since ϕ must vanish at infinity.) Hence by (6.6)

$$t(\mathbf{i}) > \tfrac{1}{2}t(\mathbf{k}).$$

We have proved the following result:

THEOREM 8. For an axially symmetric body the terminal settling speed in the axial direction is less than twice the terminal settling speed in a direction perpendicular to the axis.

7. Bounds for the terminal speeds of a slender body

We now suppose that B is convex. Let the diameter of B be $2l$ and suppose that B is inscribed in a right circular cylinder of height $2l$ and radius ϵl. If ϵ is small, we say that B is slender. We shall find bounds for the terminal speeds of B by means of theorem 5. We choose a cylindrical co-ordinate system with its origin at the centre of the cylinder and its z axis along the axis of the cylinder.

We observe that the ellipsoid of revolution

$$r^2/a^2+z^2/c^2 \leqslant 1 \tag{7.1}$$

contains the cylinder and hence also contains B, provided

$$\epsilon^2/a^2+1/c^2 = 1/l^2. \tag{7.2}$$

The ellipsoid is symmetric about the origin and hence $\lambda = 0$ for its steady falling motions. The solution of these flows is known explicitly (Oberbeck 1876; Lamb 1932, pp. 604–655; Happel & Brenner 1965, pp. 220–224). Therefore, if $\mathbf{k}$ is a vector along the axis of the cylinder and $\mathbf{i}$ is any vector perpendicular to $\mathbf{k}$, theorem 5 yields the lower bounds

$$\left.\begin{aligned} t(\mathbf{k}) &\geqslant \frac{\psi}{8\pi l(\psi^2+\epsilon^2)^{\frac{1}{2}}}\left[\frac{(2-\psi^2)}{(1-\psi^2)^{\frac{3}{2}}}\log\frac{1+(1-\psi^2)^{\frac{1}{2}}}{\psi}-\frac{1}{1-\psi^2}\right],\\ t(\mathbf{i}) &\geqslant \frac{\psi}{16\pi l(\psi^2+\epsilon^2)^{\frac{1}{2}}}\left[\frac{2-3\psi^2}{(1-\psi^2)^{\frac{3}{2}}}\log\frac{1+(1-\psi^2)^{\frac{1}{2}}}{\psi}-\frac{1}{1-\psi^2}\right],\end{aligned}\right\} \tag{7.3}$$

where we have defined $\psi = a/c$ and used the fact that by (7.2)

$$c = l(\psi^2+\epsilon^2)^{\frac{1}{2}}/\psi.$$

These lower bounds hold for any positive value of ψ, and may be maximized with respect to ψ. For small ϵ we shall use the value $\psi = \epsilon[\log(1/\epsilon)]^{\frac{1}{2}}$, which is near the maximum.

The above lower bounds hold whether B is convex or not. We now use the convexity of B to establish upper bounds. Since B is inscribed in the cylinder it is not difficult to see that it must contain the vertices of a triangle whose base is a line of length $2l$ in the z direction and whose altitude is at least ϵl. Since B is convex, it contains this triangle. We introduce a rectangular co-ordinate system with the y axis perpendicular to the triangle, the x axis pointing into the triangle and the z axis along its base. It is easily seen that the triangle contains the symmetric triangle $\frac{1}{2}\epsilon|z| \leqslant \frac{1}{2}\epsilon l - x$, $x \geqslant 0$, $y = 0$. This triangle, in turn, contains the ellipse

$$(x-\bar{a})^2/\bar{a}^2 + z^2/\bar{c}^2 \leqslant 1, \quad y = 0 \tag{7.4}$$

provided that

$$4\bar{a}/\epsilon + \bar{c}^2/l = l. \tag{7.5}$$

The solution of the flow problem for the flat ellipse is again known, since it is just an ellipsoid with one of its axes of zero length. Thus theorem 5 yields

$$\left.\begin{aligned} t(\mathbf{k}) &\leqslant \frac{(4\bar{\psi}^2+\epsilon^2)^{\frac{1}{2}}+2\bar{\psi}}{8\pi\epsilon l(1-\bar{\psi}^2)}\{(2-\bar{\psi}^2)K((1-\bar{\psi}^2)^{\frac{1}{2}})-E((1-\bar{\psi}^2)^{\frac{1}{2}})\}, \\ t(\mathbf{i}) &\leqslant \frac{(4\bar{\psi}^2+\epsilon^2)^{\frac{1}{2}}+2\bar{\psi}}{8\pi\epsilon l(1-\bar{\psi}^2)}\{(1-2\bar{\psi}^2)K((1-\bar{\psi}^2)^{\frac{1}{2}})+E((1-\bar{\psi}^2)^{\frac{1}{2}})\}, \end{aligned}\right\} \tag{7.6}$$

where K and E are the complete elliptic integrals of the first and second kinds, respectively (see Jahnke & Emde 1945, p. 73), and we have defined

$$\bar{\psi} = \bar{a}/\bar{c} = \tfrac{1}{4}\epsilon(l/\bar{c} - \bar{c}/l).$$

The direction $\mathbf{k}$ is again along the axis of the circumscribed cylinder and $\mathbf{i}$ is any direction normal to $\mathbf{k}$. We have taken for the upper bound the largest of the terminal speeds of the ellipse in directions perpendicular to $\mathbf{k}$, which occurs when $\mathbf{g}$ lies in the plane of the ellipse.

We can use the identities (see Gröbner & Hofreiter 1966, pp. 39–40)

$$K((1-\bar{\psi}^2)^{\frac{1}{2}}) = \frac{2}{\pi}K(\bar{\psi})\log\frac{4}{\bar{\psi}} - \sum_{\nu=1}^{\infty} c_\nu \bar{\psi}^{2\nu},$$

$$K(\bar{\psi}) = \tfrac{1}{2}\pi \sum_{\nu=0}^{\infty} \frac{(\frac{1}{2};1;\nu)^2}{\nu!^2}\bar{\psi}^{2\nu},$$

$$E((1-\bar{\psi}^2)^{\frac{1}{2}}) = \left(\log\frac{4}{\bar{\psi}}\right)\sum_{\nu=1}^{\infty}\frac{(\frac{1}{2};1;\nu-1)(\frac{1}{2};1;\nu)}{(\nu-1)!\,\nu!}\bar{\psi}^{2\nu} + 1 - \tfrac{1}{4}\bar{\psi}^2$$

$$-\sum_{\nu=2}^{\infty}\left(\frac{2}{1.2}+\frac{2}{3.4}+\ldots+\frac{2}{(2\nu-3)(2\nu-2)}+\frac{1}{(2\nu-1)\,2\nu}\right)\frac{(\frac{1}{2};1;\nu-1)(\frac{1}{2};1;\nu)}{(\nu-1)!\,\nu!}\bar{\psi}^{2\nu}$$

to compute these bounds. As in the case of the lower bounds, we obtain an upper bound for each positive $\bar{\psi}$ and we may find the best bound by minimizing with respect to $\bar{\psi}$. When ϵ is small, we use the approximate location

$$\bar{\psi} = \epsilon/[2\log(1/\epsilon)]$$

of the minimum.

If we make this choice of $\bar{\psi}$ and use $\psi = \epsilon(\log 1/\epsilon)^{\frac{1}{2}}$ in the lower bounds (7.3), we find that for small ϵ

$$\left.\begin{aligned} \frac{1}{4\pi l}\log\frac{1}{\epsilon} - O\left(\log\log\frac{1}{\epsilon}\right) &\leqslant t(\mathbf{k}) \\ &\leqslant \frac{1}{4\pi l}\log\frac{1}{\epsilon} + O\left(\log\log\frac{1}{\epsilon}\right), \\ \frac{1}{8\pi l}\log\frac{1}{\epsilon} - O\left(\log\log\frac{1}{\epsilon}\right) &\leqslant t(\mathbf{i}) \\ &\leqslant \frac{1}{8\pi l}\log\frac{1}{\epsilon} + O\left(\log\log\frac{1}{\epsilon}\right). \end{aligned}\right\} \tag{7.7}$$

Thus we find that

$$\left.\begin{aligned} t(\mathbf{k}) - \frac{1}{4\pi l}\log\frac{1}{\epsilon} &= O\left(\log\log\frac{1}{\epsilon}\right), \\ t(\mathbf{i}) - \frac{1}{8\pi l}\log\frac{1}{\epsilon} &= O\left(\log\log\frac{1}{\epsilon}\right), \end{aligned}\right\} \tag{7.8}$$

and hence that

$$\frac{t(\mathbf{k})}{t(\mathbf{i})} - 2 = O\left(\frac{\log\log(1/\epsilon)}{\log(1/\epsilon)}\right). \tag{7.9}$$

The last statement proves the Taylor conjecture for an arbitrary convex slender body.

We summarize our results in the following theorem.

THEOREM 9. Let B be a convex body with diameter $2l$ which can be inscribed in a circular cylinder of height $2l$ and radius $2l\epsilon$. Let $t(\mathbf{k})$ be the terminal settling speed of B in the direction of the axis of the cylinder and let $t(\mathbf{i})$ be its terminal settling speed in any direction perpendicular to the axis. Then the lower and upper bounds (7.3) and (7.6) hold for any positive values of ψ and $\bar{\psi}$. In particular, for ϵ small (7.8) and (7.9) hold, the right-hand sides being known explicit functions of ϵ. The ratio $t(\mathbf{k})/t(\mathbf{i})$ approaches two as ϵ approaches zero.

We observe that we have used the convexity of B only to prove that there is an ellipse inside B with its major axis near l and its minor axis of order $\epsilon/\log(1/\epsilon)$. The convexity condition can be replaced by the hypothesis that there is such an ellipse inside B.

Better bounds can, of course, be obtained under stronger hypotheses on B. If B is axially symmetric as well as convex, it must contain the ellipsoid of revolution

$$r^2/\bar{a}^2 + z^2/\bar{c}^2 \leqslant 1$$

when

$$4\bar{a}^2/\epsilon^2 + \bar{c}^2 = 1.$$

Since this ellipsoid contains the ellipse (7.4), its terminal speeds give better upper bounds than (7.6). However, the order of the error terms on the right is not reduced.

Still better bounds can be obtained by making stronger hypotheses about B. If we assume with Tillett (1970) that $\dot{B}$ is an axially symmetric surface with the equation

$$r = \epsilon R(z) \quad (-l \leqslant z \leqslant l),$$

where R is continuous and positive for $-l < z < l$, that the ends are rounded in the sense that $R(-l) = R(l) = 0$ and that near $z = \pm l$ the body has bounded positive curvature, it is easy to see that there are positive constants η_1 and η_2 such that B contains the ellipsoid

$$r^2/\epsilon^2\eta_1^2l^2 + z^2/l^2 \leqslant 1$$

and is contained in the ellipsoid

$$r^2/\epsilon^2\eta_2^2l^2 + z^2/l^2 \leqslant 1.$$

Thus by theorem 5 we obtain the bounds

$$\frac{1}{8\pi l}\left[\frac{2-\epsilon^2\eta_2^2}{(1-\epsilon^2\eta_2^2)^{\frac{3}{2}}}\log\frac{1+(1-\epsilon^2\eta_2^2)^{\frac{1}{2}}}{\epsilon\eta_2} - \frac{1}{1-\epsilon^2\eta_2^2}\right] \leqslant t(\mathbf{k}) \leqslant \frac{1}{8\pi l}\left[\frac{2-\epsilon^2\eta_1^2}{(1-\epsilon^2\eta_1^2)^{\frac{3}{2}}}\log\frac{1+(1-\epsilon^2\eta_1^2)^{\frac{1}{2}}}{\epsilon\eta_1} - \frac{1}{1-\epsilon^2\eta_1^2}\right],$$

$$\frac{1}{16\pi l}\left[\frac{2-3\epsilon^2\eta_2^2}{(1-\epsilon^2\eta_2^2)^{\frac{3}{2}}}\log\frac{1+(1-\epsilon^2\eta_2^2)^{\frac{1}{2}}}{\epsilon\eta_2} - \frac{1}{1-\epsilon^2\eta_2^2}\right] \leqslant t(\mathbf{i}) \leqslant \frac{1}{16\pi l}\left[\frac{2-3\epsilon^2\eta_1^2}{(1-\epsilon^2\eta_1^2)^{\frac{3}{2}}}\log\frac{1+(1-\epsilon^2\eta_1^2)^{\frac{1}{2}}}{\epsilon\eta_1} - \frac{1}{1-\epsilon^2\eta_1^2}\right].$$

For small ϵ we then see that

$$\left.\begin{aligned} t(\mathbf{k}) - \frac{1}{4\pi l}\log\frac{1}{\epsilon} &= O(1), \\ t(\mathbf{i}) - \frac{1}{8\pi l}\log\frac{1}{\epsilon} &= O(1), \end{aligned}\right\} \tag{7.10}$$

and hence
$$0 < 2 - \frac{t(\mathbf{k})}{t(\mathbf{i})} = O\left(\frac{1}{\log(1/\epsilon)}\right).$$

Since B is axially symmetric, the $\mathbf{k}$ and $\mathbf{i}$ directions lie along principal axes of the matrix $\mathbf{K}$. Thus $t(\mathbf{k})$ and $t(\mathbf{i})$ are the reciprocals of the eigenvalues of $\mathbf{K}$. Since these eigenvalues are the drag coefficients for flows in the $\mathbf{k}$ and $\mathbf{i}$ directions respectively, the formulae (7.10) verify the first term of the expansions obtained by means of slender-body theory (Batchelor 1970; Burgers 1938; Cox 1970; Taylor 1969; Tillett 1970; Tuck 1964, 1970).

The fact that the approximate relation $t(\mathbf{k})/t(\mathbf{i}) \simeq 2$ requires more than just the slenderness of B can be seen from the following example. Let B consist of the union of a sphere of radius ϵ centred at the origin and the ellipsoid of revolution whose equation in cylindrical co-ordinates is

$$\frac{r^2}{\sinh^2\delta} + \frac{z^2}{\cosh^2\delta} \leqslant 1. \tag{7.11}$$

The constant δ will be so small that $\sinh\delta < \epsilon$.

The length of B is thus $\cosh\delta$, which is greater than 1, while its width is ϵ, so that its aspect ratio is less than ϵ. We shall show that if δ is chosen so small that δ^ϵ approaches zero as $\epsilon \to 0$, then the ratio $t(\mathbf{k})/t(\mathbf{i})$ approaches one rather than two as ϵ approaches zero.

The terminal speed $\bar{t}$ of a ball of radius ϵ is known to be $1/6\pi\epsilon$ for all $\mathbf{g}$. By theorem 5, $t(\mathbf{k}) \leqslant \bar{t}$. In order to find a lower bound for $t(\mathbf{k})$, we must construct a vector field $\mathbf{v}$ which is admissible in theorem 1. The solution $\mathbf{u}$ for the fall of the ball alone in the z direction is known explicitly and can be written in the form (6.1):

$$\mathbf{u} = \operatorname{curl}[(-y\mathbf{i}+x\mathbf{j})\phi], \tag{7.12}$$

where

$$\phi = \frac{1}{24\pi\epsilon}[3\epsilon(r^2+z^2)^{-\frac{1}{2}} - \epsilon^3(r^2+z^2)^{-\frac{3}{2}}]. \tag{7.13}$$

In order to obtain a velocity field which is admissible in theorem 1 for the body B, we need to replace ϕ by a function $\tilde{\phi}$ which is constant on all of $\dot{B}$ and whose gradient is zero on $\dot{B}$. We replace the co-ordinates (z, r) by the bipolar co-ordinates (ξ, η) which are defined implicitly by the conformal mapping

$$z + ir = \sin(\xi + i\eta)$$

or

$$z = \sin\xi\cosh\eta, \quad r = \cos\xi\sinh\eta.$$

The angular variable θ is retained in the new co-ordinates. The set η = constant gives the ellipsoid

$$z^2/\cosh^2\eta + r^2/\sinh^2\eta = 1$$

with minor axis $\sinh\eta$ and foci at $r = 0$, $z = \pm 1$.

We choose any infinitely differentiable function $\tau(\alpha)$ with the properties

$$\tau = 0 \quad \text{for} \quad \alpha \leqslant 0, \quad \tau = 1 \quad \text{for} \quad \alpha \geqslant 1, \quad 0 \leqslant \tau \leqslant 1,$$

and set

$$\tilde{\phi} = \tfrac{1}{2}\bar{t} + \tau\left(\frac{2\log(\eta/\delta)}{\log(1/\delta)}\right)[\phi - \tfrac{1}{2}\bar{t}]. \tag{7.14}$$

Then the vector field $\mathbf{v} = \operatorname{curl}[(-y\mathbf{i}+x\mathbf{j})\tilde{\phi}]$ is solenoidal and has the value $\bar{t}\mathbf{k}$ on $\dot{B}$. It is therefore admissible in theorem 1. Moreover, $\mathbf{v} = \mathbf{u}$ outside the ellipsoid

$$z^2/\cosh^2\delta^{\frac{1}{2}} + r^2/\sinh^2\delta^{\frac{1}{2}} \leqslant 1.$$

Therefore $E(\mathbf{v}, \mathbf{v})$ exceeds $E(\mathbf{u}, \mathbf{u}) = \bar{t}$ by at most 2μ times the integral of $|\operatorname{grad}\mathbf{v}|^2$ inside this ellipsoid and outside the sphere $r^2+z^2 = \epsilon^2$. We now observe that $\epsilon|\phi|$, $\epsilon^2|\operatorname{grad}\phi|$, and ϵ^3 times the second derivatives of ϕ are uniformly bounded on this set, as are τ and its first two derivatives. Thus there is a constant K such that

$$\begin{aligned} E(\mathbf{v},\mathbf{v}) - E(\mathbf{u},\mathbf{u}) \leqslant K\int_{\delta}^{\delta^{\frac{1}{2}}}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} &\{\epsilon^{-6}r^2 + \epsilon^{-4} + (\epsilon^{-2}+r^2\epsilon^{-4})|\log\delta|^{-2}\eta^{-2}|\cos(\xi+i\eta)|^{-2} \\ &+ r^2\epsilon^{-2}(|\log\delta|^{-2}+|\log\delta|^{-4})\eta^{-4}|\cos(\xi+i\eta)|^{-4} \\ &+ r^2\epsilon^{-2}|\log\delta|^{-2}\eta^{-2}|\sin(\xi+i\eta)|^2|\cos(\xi+i\eta)|^{-6}\}r|\cos(\xi+i\eta)|^2\,d\xi\,d\eta. \end{aligned}$$

Using the facts that $r = \cos\xi\sinh\eta$ and that $|\cos(\xi+i\eta)|^2 = \cos^2\xi + \sinh^2\eta$, we find that there is a constant $\bar{K}$ such that

$$E(\mathbf{v},\mathbf{v}) \leqslant \bar{t}[1 + \bar{K}(\epsilon^{-1}|\log\delta|^{-1} + \epsilon^{-3}\delta + \epsilon^{-5}\delta^2)].$$

If δ is chosen so small that $\delta^{\epsilon} \to 0$ as $\epsilon \to 0$, the term in brackets approaches one, and we then see from theorems 1 and 5 that $t(\mathbf{k})/\bar{t} \to 1$ as $\epsilon \to 0$.

To obtain the same result for $t(\mathbf{i})$ we need only observe that the velocity field for the fall of the sphere in the x direction is given by

$$\mathbf{u} = \operatorname{curl}[(-z\mathbf{j}+y\mathbf{k})\phi]$$

with ϕ again defined by (7.13), and use the trial function

$$\mathbf{v} = \operatorname{curl}[(-z\mathbf{j}+y\mathbf{k})\,\tilde{\phi}]$$

with $\tilde{\phi}$ defined by (7.14) in theorem 1. Since both $t(\mathbf{k})/\tilde{t}$ and $t(\mathbf{i})/\tilde{t}$ approach one, the ratio $t(\mathbf{k})/t(\mathbf{i})$ also approaches one as $\epsilon \to 0$, as we claimed.

We see from theorem 5 that $t(\mathbf{k})/t(\mathbf{i})$ still approaches one as $\epsilon \to 0$ if B consists of the union of a ball of radius ϵ and any set that is contained in the ellipsoid (7.11), provided $\delta^{\epsilon} \to 0$ as $\epsilon \to 0$. For example, we can put a thin cylindrical tail on the sphere to obtain a slender flagellate.

The above technique is easily extended to show that attaching a sufficiently thin tail of arbitrary length to any body changes each $t(\mathbf{g})$ by an arbitrarily small amount. In particular, we can show that if two spheres of radius ϵ with their centres at $r = 0$, $z = \pm 1$ are connected by a cylindrical rod of radius δ and if $\delta^{\epsilon} \to 0$ as $\epsilon \to 0$ then $t(\mathbf{k})/t(\mathbf{i})$ will approach the value that is obtained for the spheres alone with the constraint that they cannot spin. A dilatation by a factor ϵ^{-1} shows that this ratio is the same as that for two unit spheres with their centres a distance $2\epsilon^{-1}$ apart. Since for small ϵ there is little interaction between these spheres the ratio can be shown to approach one. Thus, our dumb-bell is another example of a slender body for which $t(\mathbf{k})/t(\mathbf{i})$ does not approach two.

Finally we observe that all our bounds are obtained from flow along the principal axes of various ellipsoids. Since an ellipsoid is a non-skew body, its terminal speed is a quadratic functional of the direction. Thus upper and lower bounds for the terminal speed $t(\mathbf{k}\cos\theta+\mathbf{i}\sin\theta)$ can be obtained by adding $\cos^2\theta$ times the corresponding bound for $t(\mathbf{k})$ to $\sin^2\theta$ times the bound for $t(\mathbf{i})$. For example, we see from (7.8) that

$$t(\mathbf{k}\cos\theta+\mathbf{i}\sin\theta) - \frac{2\cos^2\theta+\sin^2\theta}{8\pi l}\log\frac{1}{\epsilon} = O\left(\log\log\frac{1}{\epsilon}\right).$$

REFERENCES

Batchelor, G. K. 1970 Slender-body theory for particles of arbitrary cross-section in Stokes flow. *J. Fluid Mech.* **44**, 419–440.

Brenner, H. 1964 The Stokes resistance of an arbitrary particle II. *Chem. Engng. Sci.* **19**, 599–624.

Burgers, J. M. 1938 On the motion of small particles of elongated form suspended in a viscous liquid. In *Second Report on Viscosity and Plasticity, Kon. Ned. Akad. Wet., Amsterdam*, **16**, 113.

Chinn, W. G. & Steenrod, N. E. 1966 *First Concepts of Topology*. New York: Random House.

Cox, R. G. 1970 The motion of long slender bodies in a viscous fluid 1. General theory. *J. Fluid Mech.* **43**, 641–660.

Diaz, J. B. 1951 Upper and lower bounds for quadratic functionals. *Proc. Symp. Spectral Theory & Differential Problems*, Stillwater, Oklahoma, pp. 279–289.

Goldman, A. J. 1966 Investigations in low Reynolds number fluid particle dynamics. Ph.D. Dissertation, Graduate Division, New York University School of Engineering.

Gröbner, W. & Hofreiter, N. 1966 *Integraltafel. Zweiter Teil: Bestimmte Integrale.* Springer.

Happel, V. & Brenner, H. 1965 *Low Reynolds Number Hydrodynamics*. Prentice-Hall.

HELMHOLTZ, H. 1968 Zur Theorie der stationären Ströme in reibenden Flüssigkeiten. *Verh. d. naturhist. med. Vereins zu Heidelberg*, **5**, 1–7. (See also *Wiss. Abh.* (collected works), **1**, 223–230.)

HILL, R. & POWER, G. 1956 Extremum principles for slow viscous flow and the approximate calculation of drag. *Quart. J. Mech. Appl. Math.* **9**, 313–319.

JAHNKE, E. & EMDE, F. 1945 *Tables of Functions*. Dover.

KEARSLEY, E. A. 1960 Bounds on the dissipation of energy in steady flow of a viscous incompressible fluid around a body rotating within a finite region. *Arch. Rat. Mech. Anal.* **5**, 347–354.

KELLER, J. B. & RUBENFELD, L. A. & MOLYNEUX, J. E. 1967 Extremum principles for slow viscous flows with applications to suspensions. *J. Fluid Mech.* **30**, 97–125.

KORTEWEG, D. J. 1883 On a general theorem of the stability of the motion of a viscous fluid. *Phil. Mag.* **16**(5), 112–118.

LAMB, H. 1932 *Hydrodynamics*. Cambridge University Press.

MIKHLIN, S. K. 1957 *Variational Methods in Mathematical Physics*. Moscow: Gostekhizdat (Trans. Pergamon, 1964).

MORAN, J. P. 1963 Line source distributions and slender-body theory. *J. Fluid Mech.* **17**, 285–304.

OBERBECK, H. A. 1876 Über stationäre Flüssigkeitsbewegungen mit Berüchsichtigung der inneren Reibung. *Crelle*, **81**, 62.

POLYA, G. & SZEGÖ, G. 1951 *Isoperimetric Inequalities in Mathematical Physics*, p. 205. Ann. Math. Studies, vol. 27.

SKALAK, R. 1970 Extensions of extremum principles for slow viscous flows. *J. Fluid Mech.* **42**, 527–548.

SYNGE, J. L. 1957 *The Hypercircle in Mathematical Physics*. Cambridge University Press.

TAYLOR, G. I. 1967 Film: *Low Reynolds number flows*. Chicago: Educational Services Inc.

TAYLOR, G. I. 1969 Motion of axisymmetric bodies in viscous fluids. In *Problems of Hydrodynamics and Continuum Mechanics in Honor of Academician L. I. Sedov*, English edn., pp. 718–724. Philadelphia: SIAM.

TILLETT, J. P. K. 1970 Axial and transverse Stokes flow past slender axisymmetric bodies. *J. Fluid Mech.* **44**, 401–417.

TUCK, E. O. 1964 Some methods for flows past slender bodies. *J. Fluid Mech.* **18**, 619.

TUCK, E. O. 1970 Toward the calculation and minimization of Stokes drag on bodies of arbitrary shape. *Proc. 3rd Aust. Conf. on Hydraulics and Fluid Mech.* 1968, pp. 29–32. Sidney: Institution of Engineers of Australia.

WEINBERGER, H. F. 1961 Variational methods in boundary-value problems. Lecture notes, University of Minnesota Engineering Bookstore.

WEINBERGER, H. F. 1972 On the steady fall of a body in a Navier–Stokes fluid. *Proc. Symp. Pure Math. Partial Differential Eqn., Providence: Amer. Math. Soc.* (in press).

WEINSTEIN, A. 1953 Generalized axially symmetric potential theory. *Bull. Am. Math. Soc.* **59**, 20–38.